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ELASTIC BENDING OF PRETWISTED BARS*

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Abstract—A method of solution is given for the elastic bending of bars with a moderate rate of initial twist. Results are obtained for an elliptical cross section showing the effect of pretwist on the bending stiffness of the corresponding untwisted bar. When the cross section is reasonably thin the effect of pretwist can be significant.

INTRODUCTION

WHEN a bar of twisted form in the unstressed state (pretwisted) is bent by end couples, it is immediately apparent that the response will be influenced by the varying orientation of the cross section, especially when the flexural rigidities (EI) about the two principal axes of the section are markedly different. The obvious first approximation is to apply locally the elementary beam formulas, relating components of moment and curvature in principal planes. This, however, disregards the helicoidal shape of the boundary, treating it locally as cylindrical or prismatic. In this paper we take account of the helicoidal shape, assuming the solution is representable as a power series in a pretwist parameter $\varkappa$. A general method of solution is presented and the first few terms of the series worked out for an elliptical cross section.

The solution shows that a moderate rate of pretwist reduces the bending stress σ_z' of elementary beam theory while introducing components of stress σ_x', σ_y', τ_{xy}', τ_{xz}', τ_{yz}' small compared to σ_z'. The local centerline curvature is correspondingly increased. For reasonably thin sections, such as those used for turbine blades, the increase in curvature for bending in the stiff (x', z) plane of the cross section can be significant—20% or more—while the increase for bending in the flexible (y', z) plane is negligible. The dominant terms of the series, showing the modifications to the stresses and curvatures of straight beam theory introduced by pretwist, are given by equations (21) and (22).

Investigations of the bending vibrations of pretwisted bars are described in references [1–13]. In analyses [1–10] it is assumed that the relation between components of moment and curvature in a principal plane of the cross section is the same locally as that for an untwisted bar. In [11–13] it is assumed only that the moment–curvature relation is linear, leaving the constant of proportionality unspecified. Experimental results are presented in

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[5–8] and [13]. In general, the analytical and experimental results indicate that a small amount of pretwist has little effect on the lowest natural frequency, but changes the higher natural frequencies appreciably when the cross section of the bar is sufficiently thin.

Zickel [14] and Maunder [15] have obtained solutions for the bending of pretwisted beams of doubly symmetric cruciform cross section, which indicate that the bending stiffness about a principal axis of the cross section is reduced by a factor involving pretwist. Using an energy method, Zickel's analysis takes into account the effect of interactions between pretwist and distortion of the cross section. Maunder obtains larger corrections which agree with his experimental results.

The static bending of a bar with one principal moment of inertia quite small compared to the other has been considered by Maunder and Reissner [16]. Using thin shallow shell theory they obtained an approximate solution for the pure bending of a pretwisted rectangular plate which is comparable to the first term of the solution obtained here.

THE BOUNDARY VALUE PROBLEM—ARBITRARY CROSS SECTION

Fixed coordinates x, y, z are placed so that z coincides with the centerline of the pretwisted bar and x and y coincide with the principal axes of the cross section at $z = 0$ (Fig. 1). All cross sections have the same shape. The cross section at z is rotated relative to the cross

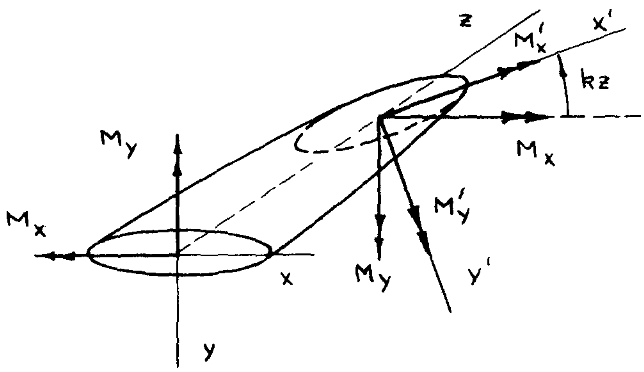

FIG. 1. Segment of a pretwisted bar.

section at the origin by an angle kz, where k is the uniform rate of pretwist. The bar is bent by a couple with components M_x and M_y about ox and oy respectively.

The well known solution for the components of stress σ_z and curvature K_x, K_y of an untwisted bar is given by

$$\sigma_z = \frac{M_x y}{I_x} - \frac{M_y x}{I_y}; \qquad K_x = \frac{M_y}{EI_y}, \quad K_y = \frac{M_x}{EI_x}. \tag{1}$$

After some exploration it appeared more advantageous to seek a stress rather than displacement solution since there is a sense in which the stress distribution is the same for each cross section, making possible a reduction of the three dimensional problem to a plane problem. It is convenient then to specify stress components, not according to curved coordinate surfaces related to the boundary shape, but in a local orthogonal cartesian system at each cross section. This system, x', y', z, is such that x' and y' coincide locally with the principal axes of the section (Fig. 1). They are related to the x, y, z system by

$$x' + iy' = e^{ikz}(x + iy). \tag{2}$$

The corresponding stress components (on an elementary parallelipiped with edges parallel to the local x', y', z directions) are written σ'_x, σ'_y, σ'_z, τ'_{xy}, τ'_{xz}, τ'_{yz}. They are related to the x, y, z components by the well known formulas of transformation from one orthogonal cartesian system to another, the scheme of direction cosines being

	x	y	z
x'	$\cos kz$	$-\sin kz$	0
y'	$\sin kz$	$\cos kz$	0
z'	0	0	1

In terms of these stress components the differential equations of equilibrium become

$$\begin{gathered}
\frac{\partial \sigma'_x}{\partial x'} + \frac{\partial \tau'_{xy}}{\partial y'} + \frac{\partial \tau'_{xz}}{\partial z} + kD_2\tau'_{xz} + k\tau'_{yz} = 0 \\
\frac{\partial \tau'_{xy}}{\partial x'} + \frac{\partial \sigma'_y}{\partial y'} + \frac{\partial \tau'_{yz}}{\partial z} + kD_2\tau'_{yz} - k\tau'_{xz} = 0 \\
\frac{\partial \tau'_{xz}}{\partial x'} + \frac{\partial \tau'_{yz}}{\partial y'} + \frac{\partial \sigma'_z}{\partial z} + kD_2\sigma'_z = 0
\end{gathered} \tag{3}$$

where

$$D_2 = -y'\frac{\partial}{\partial x'} + x'\frac{\partial}{\partial y'}.$$

To express the Beltrami–Michell compatibility conditions we introduce the operators

$$\nabla'^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z^2}; \qquad L = k^2(D_2^2 - 1) + 2k\frac{\partial}{\partial z}D_2$$

$$D_1 = 2k\left(kD_2 + \frac{\partial}{\partial z}\right)$$

and write

$$\theta' = \sigma'_x + \sigma'_y + \sigma'_z$$

The six equations are then

$$
\begin{aligned}
(1+\nu)[(\nabla'^2+L-k^2)\sigma'_x+2k^2\sigma'_y+2D_1\tau'_{xy}]+\frac{\partial^2\theta'}{\partial x'^2}&=0\\
(1+\nu)[(\nabla'^2+L-k^2)\sigma'_y+2k^2\sigma'_x-2D_1\tau'_{xy}]+\frac{\partial^2\theta'}{\partial y'^2}&=0\\
(1+\nu)(\nabla'^2+L+k^2)\sigma'_z+\left(L+k^2+\frac{\partial^2}{\partial z^2}\right)\theta'&=0\\
(1+\nu)[(\nabla'^2+L)\tau'_{yz}-D_1\tau'_{xz}]+k\left(D_2\frac{\partial}{\partial y'}-\frac{\partial}{\partial x'}\right)\theta'+\frac{\partial^2\theta'}{\partial y'\partial z}&=0\\
(1+\nu)[(\nabla'^2+L)\tau'_{xz}+D_1\tau'_{yz}]+k\left(D_2\frac{\partial}{\partial x'}+\frac{\partial}{\partial y'}\right)\theta'+\frac{\partial^2\theta'}{\partial x'\partial z}&=0\\
(1+\nu)[(\nabla'^2+L-3k^2)\tau'_{xy}+D_1(\sigma'_y-\sigma'_x)]+\frac{\partial^2\theta'}{\partial x'\partial y'}&=0
\end{aligned}
\tag{4}
$$

where ν is Poisson's ratio.

When the stress components are independent of z, equations (3) and (4) reduce to those derived by Ōkubo [17] for the torsion and tension of helicoidal rods.

For a boundary surface described by $F(x, y, z) = 0$, the direction cosines l, m, n of the normal to the surface are proportional to $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$ respectively. For the pretwisted bar, the lateral surface is described by $F'(x', y') = 0$ and the boundary conditions transform into three of the type

$$\sigma'_x l'+\tau'_{xy}m'+\tau'_{xz}n' = 0 \tag{5}$$

where the direction cosines l', m', n' are proportional to $\partial F'/\partial x'$, $\partial F'/\partial y'$, kD_2F' respectively.

REDUCTION TO A PLANE BOUNDARY VALUE PROBLEM

The components M_x, M_y of the bending couple are regarded as prescribed. The local components M'_x, M'_y defined relative to the principal axes at a cross section are the simple periodic functions of kz expressed by

$$M'_x+iM'_y = e^{ikz}(M_x+iM_y). \tag{6}$$

Since the loading is periodic, the stress components must also be periodic in kz, and in particular, it is reasonable to expect a solution for the locally defined stress components in $\cos kz$, $\sin kz$ rather than the complete Fourier series in $\cos nkz$, $\sin nkz$.

To facilitate later treatment of thin cross sections, we introduce the dimensionless coordinates ξ, η and parameters $\varkappa, \varepsilon$, defined by

$$\xi = x'/a, \quad \eta = y'/b; \qquad \varkappa = ka, \quad \varepsilon = b/a \tag{7}$$

where a and b are two representative dimensions of the cross section, b being the smaller (the minor semi-diameter for the ellipse). The locally defined stress components are taken to be of the form

$$(\sigma'_x, \sigma'_y, \sigma'_z, \tau'_{xy}, \tau'_{yz}, \tau'_{xz}) = E\cdot\mathrm{Re}[(P, Q, R, S, T, U)\,e^{-ikz}] \tag{8}$$

where

$$P = p^* + ip, \qquad Q = q^* + iq, \qquad \text{etc.}$$

and p^*, p, etc., are real functions of ξ, η, $\varkappa$, and ε.

With subscripts ξ, η indicating partial derivatives, the equations of equilibrium (3) become

$$\begin{aligned} \varepsilon P_\xi + S_\eta + \varkappa(D_4 - i\varepsilon)U + \varepsilon\varkappa T &= 0 \\ \varepsilon S_\xi + Q_\eta + \varkappa(D_4 - i\varepsilon)T - \varepsilon\varkappa U &= 0 \\ \varepsilon U_\xi + T_\eta + \varkappa(D_4 - i\varepsilon)R &= 0 \end{aligned} \tag{9}$$

where

$$D_4 = -\varepsilon^2\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}.$$

The six compatibility equations (4) become

$$\begin{aligned} (1+\nu)[\varepsilon^2 P_{\xi\xi} + P_{\eta\eta} + \varkappa^2 D_3 P + 2\varepsilon\varkappa^2 D_4(2S - iP) + 2\varepsilon^2\varkappa^2(Q - P - 2iS)] + \varepsilon^2\Phi_{\xi\xi} &= 0 \\ (1+\nu)[\varepsilon^2 Q_{\xi\xi} + Q_{\eta\eta} + \varkappa^2 D_3 Q - 2\varepsilon\varkappa^2 D_4(2S + iQ) + 2\varepsilon^2\varkappa^2(P - Q + 2iS)] + \Phi_{\eta\eta} &= 0 \\ (1+\nu)[\varepsilon^2 R_{\xi\xi} + R_{\eta\eta} + \varkappa^2(D_3 - 2i\varepsilon D_4)R] + \varkappa^2(D_3 - 2i\varepsilon D_4)\Phi &= 0 \\ (1+\nu)[\varepsilon^2 T_{\xi\xi} + T_{\eta\eta} + \varkappa^2 D_3 T - 2\varepsilon\varkappa^2 D_4(U + iT) - \varepsilon^2\varkappa^2(T - 2iU)] + \varkappa(D_4 - i\varepsilon)\Phi_\eta - \varepsilon^2\varkappa\Phi_\xi &= 0 \\ (1+\nu)[\varepsilon^2 U_{\xi\xi} + U_{\eta\eta} + \varkappa^2 D_3 U + 2\varepsilon\varkappa^2 D_4(T - iU) - \varepsilon^2\varkappa^2(U + 2iT)] + \varepsilon\varkappa(D_4 - i\varepsilon)\Phi_\xi + \varepsilon\varkappa\Phi_\eta &= 0 \\ (1+\nu)[\varepsilon^2 S_{\xi\xi} + S_{\eta\eta} + \varkappa^2 D_3 S + 2\varepsilon^2\varkappa^2(iP - iQ - 2S) + 2\varepsilon\varkappa^2 D_4(Q - P - iS)] + \varepsilon\Phi_{\xi\eta} &= 0 \end{aligned} \tag{10}$$

where

$$D_3 = D_4^2 - \varepsilon^2$$

and

$$\Phi = \varphi^* + i\varphi = P + Q + R$$

The three boundary conditions (5), with (8), become

$$(p^*, p, s^*, s, u^*, u)l' + (s^*, s, q^*, q, t^*, t)m' + (u^*, u, t^*, t, r^*, r)n' = 0 \tag{11}$$

three arising from the terms in $\cos kz$ and three from terms in $\sin kz$.

On any cross section z the stress component σ_z' is to form a couple with bending components as expressed by (6). The components of shear τ_{yz}', τ_{xz}' are to give no torsional couple or force resultants. From these conditions we obtain the relations

$$\begin{aligned} &\varepsilon^2 \iint R\eta \, d\xi \, d\eta = (M_x - iM_y)/Ea^3 &\qquad &\iint R \, d\xi \, d\eta = 0 \\ &\varepsilon \iint R\xi \, d\xi \, d\eta = -(M_x + iM_y)/Ea^3 &\qquad &\iint T \, d\xi \, d\eta = 0 \\ &\iint (\xi T - \varepsilon\eta U) \, d\xi \, d\eta = 0 &\qquad &\iint U \, d\xi \, d\eta = 0. \end{aligned} \tag{12}$$

The problem has now been reduced to the determination of the twelve real functions $p^*, p, \ldots, u^*, u$ in (8) as functions of position in the region of the ξ, η plane representing the cross section. Although the number of unknowns has been doubled, the number of independent variables has been reduced from three to two.

SERIES SOLUTION FOR MODERATE PRETWIST—ELLIPTICAL CROSS SECTION

The form of equations (9) and (10) does not encourage the expectation that a complete solution can be obtained even for the simpler cross sectional shapes. However, for a moderate rate of pretwist we may consider a solution in the form of power series in $\varkappa$ and thereby reduce the formidable boundary value problem of (9), (10), (11) and (12) to a sequence of more manageable boundary value problems. The functions $P = p^* + ip$, $Q = q^* + iq$, etc. in (8) are thus taken to be power series of the form

$$\begin{aligned} P &= \varkappa^j P_j(\xi, \eta) \\ &\vdots \\ U &= \varkappa^j U_j(\xi, \eta) \end{aligned} \tag{13}$$

where summation is understood over the repeated index j for all positive integer values including zero. The sign of $\varkappa$ is not restricted: negative values (left-handed pretwist) are permitted. But if we have a solution for a positive $\varkappa$, we can at once convert it into the solution for a negative $\varkappa$ by reflecting the bar in a mirror parallel to its axis. Inspection of the reflection shows that

$$\begin{aligned} p_j^*, p_j, q_j^*, q_j, r_j^*, r_j, s_j^*, s_j &= 0 \quad \text{for } j \text{ odd} \\ t_j^*, t_j, u_j^*, u_j &= 0 \quad \text{for } j \text{ even.} \end{aligned} \tag{14}$$

For a particular boundary shape, the solution is obtained by substituting the series (13) into (9), (10), (11) and (12), and setting the coefficients of $\varkappa^j$ equal to zero for each j. The resulting sets of equations—boundary value problems—each corresponding to a power of $\varkappa$ can be solved successively for any number of terms desired in the series (13).

The method of solution is demonstrated here for an elliptical cross section. Taking $2a$ and $2b$ as the major and minor diameters, the boundary in the ξ, η plane is the circle $\xi^2 + \eta^2 = 1$. The direction cosines l', m', n' are then proportional to $\varepsilon\xi$, η, $\varkappa(1-\varepsilon^2)\xi\eta$, and the boundary conditions (11) become

$$(p^*, p, s^*, s, u^*, u)\varepsilon\xi + (s^*, s, q^*, q, t^*, t)\eta + (u^*, u, t^*, t, r^*, r)\varkappa(1-\varepsilon^2)\xi\eta = 0 \tag{15}$$

Details of the solution are given in the Appendix. For the elliptical section, the $\varkappa$ coefficients are found to be simple polynomials in ξ and η that increase in degree with increasing powers of $\varkappa$. The resulting stress components are expressed in a manner which shows at a glance the correction terms that must be applied to the straight bar ($\varkappa = 0$) solution† (1) to account for the effects of pretwist ($\varkappa \neq 0$). The results for terms in the series

† In consequence, negative powers of ξ and η appear in some of the correction terms of (16), but are cancelled out by x' and y' in the factors representing the straight bar solution.

through $\varkappa^2$ are

$$
\begin{aligned}
\sigma_x' &= \frac{M_x'y'}{I_x}\{[3C_5^*(1-\xi^2)+C_6^*\varepsilon^2\eta^2]\varkappa^2+O(\varkappa^4)+\ldots\} \\
&\quad -\frac{M_y'x'}{I_y}\{-[C_4(1-\xi^2)+3C_3\eta^2]\varkappa^2+O(\varkappa^4)+\ldots\} \\
\sigma_y' &= \frac{M_x'y'}{I_x}\{[3C_3^*\xi^2+C_4^*\varepsilon^2(1-\eta^2)]\varkappa^2+O(\varkappa^4)+\ldots\} \\
&\quad -\frac{M_y'x'}{I_y}\{-[C_6\xi^2+3C_5(1-\eta^2)]\varkappa^2+O(\varkappa^4)+\ldots\} \\
\sigma_z' &= \frac{M_x'y'}{I_x}\{1-\tfrac{1}{2}[C_2^*(1-6\xi^2)+C_1^*\varepsilon^2(1-2\eta^2)]\varkappa^2+O(\varkappa^4)+\ldots\} \\
&\quad -\frac{M_y'x'}{I_y}\left\{1+\tfrac{1}{2}[C_1(1-2\xi^2)+C_2\varepsilon^2(1-6\eta^2)]\frac{\varkappa^2}{\varepsilon^2}+O(\varkappa^4)+\ldots\right\} \\
\tau_{xy}' &= \frac{M_x'y'}{I_x}\left\{[C_8^*(1-\xi^2)+3C_7^*\eta^2]\frac{\xi}{\eta}\varepsilon\varkappa^2+O(\varkappa^4)+\ldots\right\} \\
&\quad -\frac{M_y'x'}{I_y}\left\{-[3C_7\xi^2+C_8\varepsilon^2(1-\eta^2)]\frac{\eta}{\xi}\frac{\varkappa^2}{\varepsilon}+O(\varkappa^4)+\ldots\right\} \\
\tau_{xz}' &= \frac{M_x'y'}{I_x}\left\{\frac{(1-\varepsilon^2)}{2(1+\nu)(3+\varepsilon^2)}\frac{(1-\xi^2-3\eta^2)}{\eta}\varepsilon\varkappa+O(\varkappa^3)+\ldots\right\} \\
&\quad -\frac{M_y'x'}{I_y}\left\{-\frac{(1-\varepsilon^2)}{(1+3\varepsilon^2)}\left[\frac{\nu}{(1+\nu)}+3\varepsilon^2\right]\eta\frac{\varkappa}{\varepsilon}+O(\varkappa^3)+\ldots\right\} \\
\tau_{yz}' &= \frac{M_x'y'}{I_x}\left\{-\frac{(1-\varepsilon^2)}{(3+\varepsilon^2)}\left[3+\frac{\nu}{(1+\nu)}\varepsilon^2\right]\xi\varkappa+O(\varkappa^3)+\ldots\right\} \\
&\quad -\frac{M_y'x'}{I_y}\left\{\frac{(1-\varepsilon^2)}{2(1+\nu)(1+3\varepsilon^2)}\frac{(1-3\xi^2-\eta^2)}{\varepsilon}\varkappa+O(\varkappa^3)+\ldots\right\}
\end{aligned}
\tag{16}
$$

where $I_x = \pi ab^3/4$, $I_y = \pi a^3b/4$,

$$C_i = \frac{(1-\varepsilon^2)(a_i+b_i\varepsilon^2+c_i\varepsilon^4+d_i\varepsilon^6+e_i\varepsilon^8)}{3(1+\nu)(3+\varepsilon^2)(1+3\varepsilon^2)(1+2\varepsilon^2+5\varepsilon^4)}; \qquad i = 1, 2, \ldots, 8$$

$$C_i^* = \frac{(1-\varepsilon^2)(e_i+d_i\varepsilon^2+c_i\varepsilon^4+b_i\varepsilon^6+a_i\varepsilon^8)}{3(1+\nu)(3+\varepsilon^2)(1+3\varepsilon^2)(5+2\varepsilon^2+\varepsilon^4)}; \qquad i = 1, 2, 3, 5 \text{ and } 7$$

$$= \frac{(1-\varepsilon^2)(d_i+c_i\varepsilon^2+b_i\varepsilon^4+a_i\varepsilon^6)}{3(1+\nu)(3+\varepsilon^2)(1+3\varepsilon^2)(5+2\varepsilon^2+\varepsilon^4)}; \qquad i = 4, 6 \text{ and } 8$$

and

$$a_1 = 3\nu(1+\nu) \qquad b_1 = 28+23\nu+28\nu^2 \qquad c_1 = 68+131\nu+66\nu^2$$
$$a_2 = 3(2-\nu^2) \qquad b_2 = 4-7\nu-28\nu^2 \qquad c_2 = 8-93\nu-66\nu^2$$
$$a_3 = -3\nu \qquad b_3 = -2(2+11\nu) \qquad c_3 = -(73+28\nu)$$
$$a_4 = 3(1+2\nu) \qquad b_4 = 2(10+19\nu) \qquad c_4 = 111+66\nu$$
$$a_5 = 3(1-\nu) \qquad b_5 = -(9+10\nu) \qquad c_5 = -10$$
$$a_6 = -18(1-\nu) \qquad b_6 = 3(15+22\nu) \qquad c_6 = 2(22+19\nu)$$
$$a_7 = -3\nu \qquad b_7 = -(11+10\nu) \qquad c_7 = -14$$
$$a_8 = 3(1-2\nu) \qquad b_8 = -2(1+19\nu) \qquad c_8 = -3(23+22\nu)$$
$$d_1 = 164+153\nu+28\nu^2 \qquad e_1 = 3(20+14\nu+\nu^2)$$
$$d_2 = -(52+97\nu+28\nu^2) \qquad e_2 = -3(10+9\nu+\nu^2)$$
$$d_3 = 2(7+19\nu) \qquad e_3 = 15(1+\nu)$$
$$d_4 = 6(7+3\nu) \qquad e_4 = 0$$
$$d_5 = 3+10\nu \qquad e_5 = -3(1-\nu)$$
$$d_6 = 3(3+2\nu) \qquad e_6 = 0$$
$$d_7 = 29+10\nu \qquad e_7 = 3(4+\nu)$$
$$d_8 = -6(2+3\nu) \qquad e_8 = 0.$$

For the case $\varkappa = 0$, the straight bar, and for $\varepsilon = 1$, the bar with circular cross section, all the correction terms vanish, as would be expected.

CENTERLINE CURVATURE

Components of centerline curvature may now be determined using the stress solution (16). For small displacements u and v, the components of centerline curvature in the x, z and y, z planes are approximated by

$$\begin{aligned} K_x &= \frac{\partial^2 u}{\partial z^2} = \frac{2(1+\nu)}{E}\frac{\partial \tau_{xz}}{\partial z} - \frac{1}{E}\frac{\partial}{\partial x}[\sigma_z - \nu(\sigma_x+\sigma_y)] \\ K_y &= -\frac{\partial^2 v}{\partial z^2} = -\frac{2(1+\nu)}{E}\frac{\partial \tau_{yz}}{\partial z} + \frac{1}{E}\frac{\partial}{\partial y}[\sigma_z - \nu(\sigma_x+\sigma_y)]. \end{aligned} \tag{17}$$

Local components K'_x and K'_y defined relative to the principal axes at a cross section, are related to K_x and K_y by

$$K'_x + iK'_y = \mathrm{e}^{-ikz}(K_x + iK_y). \tag{18}$$

Using (2) and the appropriate stress transformation, the local components of curvature are expressed in terms of local components of stress by

$$K'_x = \frac{2(1+\nu)}{E}\left(\frac{\partial\tau'_{xz}}{\partial z}+k\tau'_{yz}\right)-\frac{1}{E}\frac{\partial}{\partial x'}[\sigma'_z-\nu(\sigma'_x+\sigma'_y)]$$
$$K'_y = -\frac{2(1+\nu)}{E}\left(\frac{\partial\tau'_{yz}}{\partial z}-k\tau'_{xz}\right)+\frac{1}{E}\frac{\partial}{\partial y'}[\sigma'_z-\nu(\sigma'_x+\sigma'_y)]. \quad (19)$$

For the elliptical cross section the components of centerline curvature obtained by substitution of (16) into (19) are

$$K'_x = \frac{M'_y}{EI_y}\left\{1+\frac{C_1}{2}\frac{\varkappa^2}{\varepsilon^2}+\left[\frac{C_2}{2}+\nu C_4+3\nu C_5-\frac{4(1-\varepsilon^4)}{(3+\varepsilon^2)(1+3\varepsilon^2)}\right]\varkappa^2+O(\varkappa^4)+\ldots\right\}$$
$$K'_y = \frac{M'_x}{EI_x}\left\{1-\left[\frac{C_2^*}{2}+3\nu C_5^*+\left(\frac{C_1^*}{2}+\nu C_4^*\right)\varepsilon^2-\frac{4(1-\varepsilon^4)\varepsilon^2}{(3+\varepsilon^2)(1+3\varepsilon^2)}\right]\varkappa^2+O(\varkappa^4)+\ldots\right\}. \quad (20)$$

SUMMARY OF RESULTS

It will be observed that the constants C_i and C_i^* defined below equations (16) may be represented as power series in ε starting with ε to the zero power. Since it is known from experiments that the effect of pretwist is most pronounced for thin sections, we may simplify the solutions (16) and (20) without great loss of generality by limiting ε to small values. Expanding C_i and C_i^* in power series in ε, and neglecting terms of order ε^2 compared to 1, the predominant terms in the $\varkappa$ series ($\varkappa^2$ small compared to 1) for stress components are found to be

$$\sigma'_z = \frac{M'_x y'}{I_x}-\frac{M'_y x'}{I_y}\left[1+\frac{\nu}{6}(1-2\xi^2)\frac{\varkappa^2}{\varepsilon^2}\right]$$
$$\tau'_{yz} = -\frac{M'_y x'}{I_y}\frac{(1-3\xi^2-\eta^2)}{2(1+\nu)\xi}\varkappa \quad (21)$$

and for curvature components

$$K'_x = \frac{M'_y}{EI_y}\left[1+\frac{\nu}{6}\frac{\varkappa^2}{\varepsilon^2}\right]; \qquad K'_y = \frac{M'_x}{EI_x}. \quad (22)$$

Summarizing the results, we find that for moderate rates of pretwist and reasonably thin sections, the classical solution for a straight bar must be modified for bending in the stiff (x', z) plane of the cross section but is relatively unaffected for bending in the flexible plane. For an elliptical cross section with $\varepsilon = 0{\cdot}1$, $\varkappa = 0{\cdot}2$, $\nu = 0{\cdot}3$, the largest stress (σ'_z at $\xi = \pm 1$) is reduced by 20% while the curvature K'_x is increased by 20%. Thus for dimensions in the range of interest for turbine blades, for instance, the effect of pretwist is significant. This tends to bear out findings of previous analyses and experiments that the lowest natural frequency, corresponding roughly to bending in the flexible plane, is essentially unchanged by a moderate pretwist. But for the higher natural frequencies it would appear that the straight bar relations assumed in references [1–10] could lead to significant errors.

The stress components (21) and curvature components (22) may be compared with results obtained by Maunder and Reissner [16] for the bending of a shallow hyperbolic paraboloidal shell—essentially a pretwisted plate. They found the sheet stress and centerline curvature at the origin ($z = 0$), due to bending in the stiff (x, z) plane, to be

$$\frac{1}{2b}N_{xx} = -\frac{M_y x}{I_x}\left[1+\frac{\nu}{15}(3-5\xi^2)\frac{\varkappa^2}{\varepsilon^2}\right]$$

$$K_x = \frac{M_y}{EI_y}\left[1+\frac{\nu}{5}\frac{\varkappa^2}{\varepsilon^2}\right]$$

while the curvature and stresses due to bending in the flexible (y, z) plane are unaffected by pretwist. Thus the leading terms of the series (16) and (20) obtained here are in substantial agreement with the Maunder and Reissner results, the small differences being due no doubt to the difference in cross sectional shapes treated.

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APPENDIX—SOLUTION FOR COEFFICIENTS OF THE $\varkappa$ SERIES

Zeroth boundary value problem ($\varkappa^0$)

The differential equations and boundary conditions for the elliptical cross section are given by (9), (10), (12) and (15). Substituting the series (13) into these equations, and setting the coefficients for each power of $\varkappa$ equal to zero, we obtain sets of equations relating $P_j, Q_j, \ldots U_j$.

The set of equations obtained from the coefficients of $\varkappa^\circ$ is

From (9):

$$\begin{aligned} \varepsilon P_{0,\xi} + S_{0,\eta} &= 0 \\ \varepsilon S_{0,\xi} + Q_{0,\eta} &= 0. \end{aligned} \tag{A1}$$

From (10):

$$\begin{aligned} (1+\nu)[\varepsilon^2 P_{0,\xi\xi} + P_{0,\eta\eta}] + \varepsilon^2 \Phi_{0,\xi\xi} &= 0 \\ (1+\nu)[\varepsilon^2 Q_{0,\xi\xi} + Q_{0,\eta\eta}] + \Phi_{0,\eta\eta} &= 0 \\ \varepsilon^2 R_{0,\xi\xi} + R_{0,\eta\eta} &= 0 \\ (1+\nu)[\varepsilon^2 S_{0,\xi\xi} + S_{0,\eta\eta}] + \varepsilon \Phi_{0,\xi\eta} &= 0. \end{aligned} \tag{A2}$$

From (15):

$$\begin{aligned} \varepsilon \xi P_0 + \eta S_0 &= 0 \\ \varepsilon \xi S_0 + \eta Q_0 &= 0. \end{aligned} \tag{A3}$$

Equations (A2) can be combined with (A1) and rewritten in the form

$$\begin{aligned} \varepsilon^2 R_{0,\xi\xi} + R_{0,\eta\eta} &= 0 \\ \varepsilon^2 P_{0,\xi\xi} + P_{0,\eta\eta} - \frac{1}{\nu} R_{0,\eta\eta} &= 0 \\ \varepsilon^2 Q_{0,\xi\xi} + Q_{0,\eta\eta} + \frac{1}{\nu} R_{0,\eta\eta} &= 0 \\ \varepsilon^2 S_{0,\xi\xi} + S_{0,\eta\eta} + \frac{\varepsilon}{\nu} R_{0,\xi\xi} &= 0 \end{aligned} \tag{A4}$$

Now, assuming a polynomial solution in ξ and η, we express P_0, Q_0, R_0 and S_0 as sums of polynomials

$$\begin{aligned} P_0 &= P_{oij}(\nu, \varepsilon)\eta^i \xi^j \\ &\vdots \\ S_0 &= S_{oij}(\nu, \varepsilon)\eta^i \xi^j \qquad i, j = 0, 1, 2, \ldots \end{aligned} \tag{A5}$$

To simplify the solution, we consider only the component of bending moment $M_x (M_y = 0)$. Then, from symmetry of the cross section about ξ and η, it can be seen that

$$\begin{aligned} p^*, q^*, r^*, s &= 0 \quad \text{for } i \text{ even}, j \text{ odd} \\ p, q, r, s^* &= 0 \quad \text{for } i \text{ odd}, j \text{ even} \end{aligned} \tag{A6}$$

Substituting (A5) into (A4) and (A1), the polynomial forms of p_o, q_o, r_o, s_o that satisfy (A1) and (A2) are found to be

$$r_0 = \text{Re}\left[\sum_{j\ odd} r_{oj}\zeta^j\right]$$

$$\begin{aligned} p_0 &= -\text{Re}\left[\sum_{j\ odd} s_{oj}\zeta^j\right] - \frac{3}{v}r_{03}\varepsilon^2\xi\eta^2 - \frac{10}{v}r_{05}\xi(\varepsilon^2\xi^2\eta^2 - \varepsilon^4\eta^4) + \ldots \\ q_0 &= \text{Re}\left[\sum_{j\ odd} s_{oj}\zeta^j\right] - (s_{01} - q_{01})\xi + \frac{1}{v}r_{03}\xi^3 + \frac{1}{v}r_{05}\xi(\xi^4 - 5\varepsilon^4\eta^4) + \ldots \\ s_0 &= \text{Im}\left[\sum_{j\ odd} s_{oj}\zeta^j\right] + \frac{1}{v}r_{03}\varepsilon^3\eta^3 + \frac{2}{v}r_{05}\varepsilon\eta(5\varepsilon^2\xi^2\eta^2 - \varepsilon^4\eta^4) + \ldots \end{aligned} \tag{A7}$$

where $\zeta = \xi + i\varepsilon\eta$. Similar sets of polynomials are obtained for $p_o^*, q_o^*, r_o^*, s_o^*$. It should be noted at this point that the solution has not yet been restricted to the elliptical cross section, only to a doubly symmetric section, and that is not essential to the method. We might reasonably expect, therefore, that the solution for other cross sectional shapes described by simple polynomials in ξ and η would be no more difficult.

The coefficients of the polynomials (A7) are obtained from the boundary conditions. Putting (A7) into (A3) and replacing η^2 by $1-\xi^2$, the set of equations obtained from the coefficients of each power of ξ, written in matrix form, is

$$A\mathbf{r} = 0 \tag{A8}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ & & a_{33} & a_{34} & \cdots \\ & & a_{43} & a_{44} & \cdots \\ & 0 & & & \cdots \\ & & & & \cdots \\ & & & & \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} s_{01} \\ q_{01} \\ s_{03} \\ r_{03} \\ s_{05} \\ r_{05} \\ \vdots \end{bmatrix}$$

and the a_{ij} are functions of v and ε only. It can be seen from the derivation of (A8) that the 2×2 blocks of elements on the diagonal of A are nonsingular. Therefore A is nonsingular and the solution of (A8) is

$$\mathbf{r} = 0 \tag{A9}$$

The only remaining nonzero coefficient is r_{01}, which is found from the conditions (12) ($M_y = 0$) to be

$$r_{01} = -\frac{4M_x}{\pi E a^3 \varepsilon} \tag{A10}$$

The solution to the imaginary parts of (A1), (A2) and (A3) thus becomes

$$\begin{aligned} p_o, q_o, s_o &= 0 \\ r_o &= -\frac{M_x x'}{EI_y} \end{aligned} \tag{A11}$$

Similarly, the solution to the real parts of (A1), (A2) and (A3) is found to be

$$\begin{aligned} p_o^*, q_o^*, s_o^* &= 0 \\ r_o^* &= \frac{M_x y'}{EI_x} \end{aligned} \tag{A12}$$

With a solution for $M_x(M_y = 0)$, the solution for $M_y(M_x = 0)$ can be obtained by replacing kz in (8) by $kz + \frac{1}{2}\pi$.

Of course the results in (A11) and (A12) could have been deduced immediately from the straight bar solution ($\varkappa = 0$) and verified by substitution into (A1), (A2) and (A3). The purpose of solving the equations here is to demonstrate the method and show the solution is unique.

First boundary value problem ($\varkappa^1$)

The set of equations obtained by setting the coefficients of $\varkappa^1$ equal to zero is

From (9):

$$\varepsilon U_{1,\xi} + T_{1,\eta} = -(1-\varepsilon^2)(\xi - i\varepsilon\eta)B. \tag{A13}$$

From (10):

$$\begin{aligned} (1+\nu)[\varepsilon^2 T_{1,\xi\xi} + T_{1,\eta\eta}] &= i\varepsilon(1-\varepsilon^2)B \\ (1+\nu)[\varepsilon^2 U_{1,\xi\xi} + U_{1,\eta\eta}] &= -\varepsilon(1-\varepsilon^2)B. \end{aligned} \tag{A14}$$

From (15):

$$\varepsilon\xi U_1 + \eta T_1 = -(1-\varepsilon^2)(\eta - i\varepsilon\xi)\xi\eta B \tag{A15}$$

where

$$B = \frac{4M_x}{\pi E a^3 \varepsilon^2}.$$

Again assuming a polynomial form of solution, T_1 and U_1 are expressed as

$$\begin{aligned} T_1 &= T_{1ij}(\nu, \varepsilon)\eta^i\xi^j \\ U_1 &= U_{1ij}(\nu, \varepsilon)\eta^i\xi^j \qquad i,j = 0, 1, 2, \ldots \end{aligned} \tag{A16}$$

and it is noted from symmetry of the cross section that

$$\begin{aligned} t, u^* &= 0 \quad \text{for } i \text{ odd}, j \text{ odd} \\ t^*, u &= 0 \quad \text{for } i \text{ even}, j \text{ even}. \end{aligned} \tag{A17}$$

By substitution of (A16) into (A13) and (A14) the polynomial forms of t_1 and u_1 are found to be

$$\begin{aligned} t_1 &= \frac{\varepsilon(1-\varepsilon^2)}{2(1+\nu)}B\eta^2 + \mathrm{Re}\left[\sum_{j \text{ even}} t_{oj}\zeta^j\right] \\ u_1 &= \frac{\nu(1-\varepsilon^2)}{(1+\nu)}B\xi\eta + \mathrm{Im}\left[\sum_{j \text{ even}} t_{oj}\zeta^j\right] \end{aligned} \tag{A18}$$

where $\zeta = \xi + i\varepsilon\eta$.

Substituting (A18) into the boundary condition (A15), and replacing η^2 by $1-\xi^2$, the set of equations obtained from the coefficients of each power of ξ, written in matrix form, is

$$A\mathbf{t} = \mathbf{b} \tag{A19}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ & a_{22} & a_{23} & \cdots \\ & & a_{33} & \cdots \\ & 0 & & \ddots \\ & & & \end{bmatrix}, \qquad \mathbf{t} = \begin{bmatrix} t_{00} \\ t_{02} \\ t_{04} \\ \\ \vdots \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

and the a_{ij} and b_i are known ($b_i = 0$ for $i > 2$). Since the diagonal elements of A are non-singular, the solution to (A19) is

$$\mathbf{t} = A^{-1}\mathbf{b} \tag{A20}$$

where the form of A^{-1} is

$$A^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots \\ & b_{22} & b_{23} & \cdots \\ & & b_{33} & \cdots \\ & 0 & & \ddots \end{bmatrix}$$

Thus the solution (A20) of (A19) is

$$\begin{aligned} t_{00} &= b_{11}b_1 + b_{12}b_2 \\ t_{02} &= b_{22}b_2 \\ t_{04}, t_{06}, t_{08}, \ldots &= 0. \end{aligned} \tag{A21}$$

Since A is upper triangular, the b_{ij} are readily determined from the first few a_{ij}. Thus

$$b_{11} = \frac{1}{a_{11}} = 1; \qquad b_{22} = \frac{1}{a_{22}} = \frac{1}{(1+3\varepsilon^2)};$$

$$b_{12} = -\frac{a_{12}}{a_{11}a_{22}} = \frac{\varepsilon^2}{(1+3\varepsilon^2)}$$

and

$$b_1 = -\frac{\varepsilon(1-\varepsilon^2)}{2(1+\nu)}B, \quad b_2 = \frac{3\varepsilon(1-\varepsilon^2)}{2(1+\nu)}B.$$

The resulting solution of the imaginary parts of (A13), (A14), and (A15) is

$$\begin{aligned} t_1 &= -\frac{\varepsilon(1-\varepsilon^2)B}{2(1+\nu)(1+3\varepsilon^2)}(1-3\xi^2-\eta^2); \\ u_1 &= \left[\frac{\nu}{(1+\nu)}+3\varepsilon^2\right]\frac{(1-\varepsilon^2)B}{(1+3\varepsilon^2)}\xi\eta. \end{aligned} \tag{A22}$$

Similarly, the solution of the real parts of (A13), (A14), and (A15) can be found to be

$$\begin{aligned} t_1^* &= -\left[3+\frac{\nu}{(1+\nu)}\varepsilon^2\right]\frac{(1-\varepsilon^2)}{(3+\varepsilon^2)}B\xi\eta \\ u_1^* &= \frac{\varepsilon(1-\varepsilon^2)B}{2(1+\nu)(3+\varepsilon^2)}(1-\xi^2-3\eta^2). \end{aligned} \tag{A23}$$

It is easily verified that (A22) and (A23) also satisfy the conditions (12) on each cross section.

Solution of the boundary value problems associated with the higher powers of $\varkappa$ continues in a similar fashion, the difficulty encountered being no greater than for the first two boundary value problems shown here. The sets of equations are similar, differing only in the nonhomogeneous terms.

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Абстракт—Дается решение для упругого изгиба балок со средней скоростью началрного кручения. Получаются результаты для эллиптического поперечного разраза, показывающие эффект предварительного кручения на жесткость изгиба соответсвенной нескрученной балки. Если поперечный разрез является умеренно тонким, эффект передварительного кручения может оказаться значительным.